Non-Gaussian fluctuations in electromagnetic radiation scattered by random phase screen. I. Theory

1975 J. Phys. A: Math. Gen. 8 369

(http://iopscience.iop.org/0305-4470/8/3/010)

J. Phys. A: Math. Gen., Vol. 8, No. 3, 1975. Printed in Great Britain. © 1975

Non-Gaussian fluctuations in electromagnetic radiation scattered by a random phase screen. I. Theory

E Jakeman and P N Pusey
Royal Radar Establishment, Malvern, Worcestershire WR14 3PS, UK

Received 14 August 1974

Abstract. The statistical, spatial and temporal coherence properties of electromagnetic radiation scattered into the far field by a deep random phase screen are investigated. It is shown that significant departures from Gaussian behaviour can occur even when the phase correlation length is much smaller than the dimensions of the scattering region—a situation in which the central limit theorem might be expected to apply. Formulae are derived relating these departures to elementary properties of the scattering structure, which may therefore be determined by measurement of the fluctuations in the scattered radiation. Application of the results to scattering from very rough surfaces is discussed.

1. Introduction

It is well known that an electromagnetic field which is composed of independent randomly phased contributions from a large number of scattering centres will be Gaussian-distributed by virtue of the central limit theorem (see for example Davenport and Root 1958, chap 5). The statistical properties of such a field contain no information other than that there are many scatterers. However, if the number N of scatterers is small, so that the central limit theorem cannot be applied, then the statistics will deviate from Gaussian by an amount depending on N and other parameters characterizing the scattering process. In previous publications (Jakeman and Pusey 1973a, b) we reported some preliminary theoretical and experimental results of an investigation of the statistical properties of light scattered by a random phase screen, ie a system which retards the phase of an incident electromagnetic field by a randomly varying, position-dependent amount. When the mean square phase deviation is equivalent to path differences of the order of a wavelength of the radiation or more (deep phase screen) each phase 'correlation area' can be thought of as giving an independent randomly phased contribution to the far field. Our results were valid even when the scattering region extended over only a small number of correlation areas so that, as expected, they showed significant departures from Gaussian statistics depending on the phase correlation length ξ, mean square deviation $\overline{\phi^2}$ and on W_0, the size of the scattering region. In the light of earlier work (Deutsch and Keating 1969) the phase screen model was used to characterize 'dynamic scattering' exhibited by a thin layer of nematic liquid crystal under the influence of an applied electric field (Heilmeier *et al* 1968). The spatial scale of refractive index fluctuations in this system was expected to be of the order of a few microns so that a laser beam could be focused down to illuminate an area sufficiently small for non-Gaussian effects to be important. Experimental observations (Jakeman and Pusey 1973b) were in good agreement with the theoretical predictions based on a joint-Gaussian model for the phase fluctuations and enabled the parameters ξ and $\overline{\phi^2}$ of the model to be determined.

Although this previous work was stimulated by an interest in the properties of light scattered by liquid crystals, the theoretical approach used applies equally well to the scattering of radiation of other wavelengths. Moreover, many naturally occurring phenomena other than dynamic scattering are the result of electromagnetic radiation passing through a random phase screen. Familiar examples are the twinkling of star-light, the fading of radio signals due to fluctuations in the ionosphere, and the randomly varying pattern which can be observed on the floor of a shallow pond or swimming pool when the water surface is disturbed. Perhaps more importantly, any rough surface behaves as a deep random phase screen when illuminated with sufficiently short-wave radiation.

It seems likely, therefore, that a study of the statistics and coherence properties of electromagnetic radiation scattered by a deep random phase screen in the non-Gaussian regime (when the scattering region is comparable in size to the spatial correlation length of the phase fluctuations) might prove to have application in a number of fields. We have already mentioned N, the effective number of scattering centres, as a parameter governing the size of the non-Gaussian effects, but it is not clear what other parameters are important in this context nor whether the statistics of the scattered radiation can in general be usefully related to these parameters. It is appropriate at this point, therefore, to outline a simple intuitive picture of the origin of non-Gaussian fluctuations in radiation scattered by a deep random phase screen and to establish qualitatively the type of information which may be obtained by their measurement.

Consider, then, the back-scattered radiation at an angle θ (figure 1) from a perfectly reflecting very rough surface (path differences introduced of the order of a wavelength

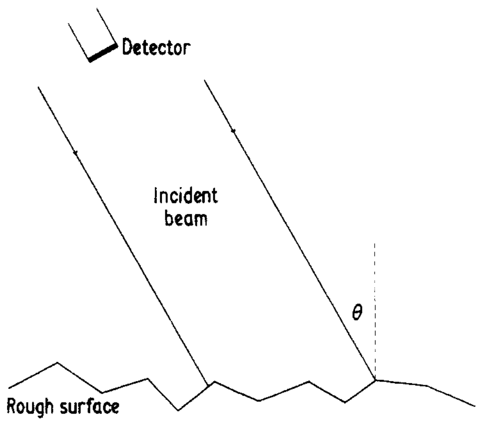

Figure 1. Simple back-scattering geometry.

or greater). The scattering region can be thought of as containing N independently oriented facets. For simplicity we shall assume that the surface is illuminated by a plane parallel beam of coherent radiation but that the detector area is larger than the coherence area defined in the far field by the scattering region (see for example Mandel and Wolf 1965) so that intensities add. Neglecting diffraction effects, the intensity measured by a square-law envelope detector will be proportional to the number of facets $n(\theta)$ normal to the direction θ:

$$I = \frac{C}{N} n(\theta) \tag{1}$$

where C is independent of N. Suppose, now, that the surface is translated through the beam; the intensity will fluctuate with mean and mean square values given by

$$\langle I \rangle = \frac{C}{N} \langle n(\theta) \rangle \tag{2}$$

and

$$\frac{\langle I^2 \rangle}{\langle I \rangle^2} = \frac{\langle n^2(\theta) \rangle}{\langle n(\theta) \rangle^2} \tag{3}$$

where the angular brackets denote ensemble or time averages. If the facet orientations are independent then it is reasonable to assume that the distribution of $n(\theta)$ will be Poisson-like about the mean number normal to the direction θ. Thus we assume that

$$\langle n^2(\theta) \rangle - \langle n(\theta) \rangle^2 = \langle n(\theta) \rangle.$$

Equation (3) then takes the form

$$\frac{\langle I^2 \rangle}{\langle I \rangle^2} = 1 + \frac{1}{\langle n(\theta) \rangle}. \tag{4}$$

Now $\langle n(\theta) \rangle$ is just the total number of facets N times the probability $p(\theta)$ of finding one normal to the direction θ so that (2) and (4) may be written

$$\langle I \rangle = Cp(\theta) \tag{5}$$

$$\frac{\langle I^2 \rangle}{\langle I \rangle^2} = 1 + \frac{1}{Np(\theta)}. \tag{6}$$

The second term on the right-hand side of equation (6) is a 'non-Gaussian' effect due to the finite number of facets and vanishes as $N \to \infty$. In this limit the intensity is constant, ie $\langle I^2 \rangle = \langle I \rangle^2$, since in the simple case considered here incoherent detection has averaged out the Gaussian fluctuations. In general when N is finite, (5) and (6) exhibit several interesting features. First it is evident that $\langle I \rangle$ is independent of the size of the scattering region whereas the second moment (6) depends on both the surface slope distribution ($\sim p(\theta)$) and the number of facets. Secondly, since the chance of finding a very large surface slope will usually be small, the non-Gaussian term in (6) may well be significant at large θ even when N is large. This apparent contradiction of the central limit theorem is a consequence of the fact that although there may be many scattering centres in the target area, only a small fraction contribute to the intensity in the direction θ at any one time. This can be expressed in a different way by saying that convergence to Gaussian statistics as predicted by the central limit theorem is slow due to fluctuations in the cross section of each scattering centre. A further property of the second-order statistic (6) is that it is independent of the absolute magnitude of the intensity and its measurement could therefore provide an attractive way of determining N and $p(\theta)$.

The intuitive approach described above gives useful insight into the way in which non-Gaussian fluctuations arise and into what parameters govern their magnitude.

However, it only provides a qualitative picture of the origin of certain statistical properties since we have made many simplifying assumptions and approximations. For example, we have neglected diffraction effects, variation in the size of facets and the consequential fluctuations in N. In this paper, therefore, we carry out a more quantitative investigation of the statistical and coherence properties of electromagnetic radiation scattered by a deep random phase screen. The companion paper II reports experimental measurements made on the liquid crystal system, referred to earlier, and compares the results obtained with the formulae derived here.

Two separate theoretical methods are described in §§2 and 3. The first is a direct analytical approach based on a joint-Gaussian model for the phase fluctuations and certain other well defined assumptions and approximations. The second is more closely related to the empirical method used above, being based on the division of the scattering region into a number of independent 'micro-areas'. Both techniques lead to similar expressions for the first- and second-order intensity moments provided that certain reasonable parameter identifications are made; preliminary results for these quantities have already been published (Jakeman and Pusey 1973a, b). The temporal and spatial intensity correlation functions can only be extracted with some difficulty in the analytical case, however, whilst the micro-area approach proves to be more tractable both for the evaluation of these and the higher-order statistical properties of the intensity. The two methods are outlined in Jakeman (1974) and repetition is avoided where possible although some details of the calculations are included in an appendix. The results obtained in §§2 and 3 are discussed and compared in the following section. In §5 application of the formulae to scattering from rough surfaces is briefly discussed.

2. Direct analytical approach

We shall consider the experimental arrangement shown in figure 2 in which a collimated beam of electromagnetic radiation is incident on a phase screen of negligible thickness and the forward-scattered radiation is detected in the far field (Fraunhofer region) by a square-law envelope detector whose axis makes an angle θ with the direction of incidence. The positive frequency part of the electric field at the detector point $\boldsymbol{r} \equiv (R, \theta, z) \equiv (\boldsymbol{R}, z)$ is given by (Jakeman 1974)

$$\mathscr{E}^+(\boldsymbol{r}, t) = E_0 e^{-i\omega t} \int_{-\infty}^{+\infty} d^2r' \exp(ik|\boldsymbol{r}' - \boldsymbol{r}|) \exp(i\phi(\boldsymbol{r}', t)) \exp(-r'^2/W_0^2) \qquad (7)$$

where $k = \omega/c$ is the wavevector of the light, $\phi(\boldsymbol{r}', t)$ is a randomly fluctuating phase variable introduced by the screen in the $z = 0$ plane and E_0 is a constant. We have

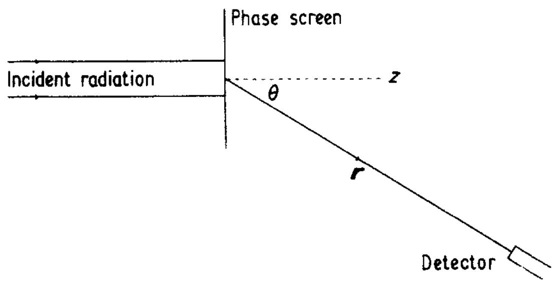

Figure 2. Experimental arrangement for detection of radiation scattered by a phase screen.

assumed that the incident beam has a Gaussian intensity profile of width W_0 mainly for mathematical convenience but this would be the normal situation in laser light scattering experiments for example (see paper II). In the far field

$$|\boldsymbol{r}'-\boldsymbol{r}| \simeq r-\boldsymbol{R}\,.\,\boldsymbol{r}'/r$$

where $r = (R^2+z^2)^{1/2}$ and (7) becomes

$$\mathscr{E}^+(\boldsymbol{r},t) = E_0 \exp[\mathrm{i}(kr-\omega t)] \int_{-\infty}^{+\infty} \mathrm{d}^2r' \exp(-\mathrm{i}k\boldsymbol{R}\,.\,\boldsymbol{r}'/r \exp(\mathrm{i}\phi(r',t)) \exp(-r'^2/W_0^2). \tag{8}$$

2.1. Statistics

Defining the intensity

$$I(\boldsymbol{r},t) = \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r},t) \tag{9}$$

and assuming that ϕ is Gaussian distributed so that

$$\langle\exp(-\mathrm{i}\Sigma_j\phi(r_j,t))\rangle = \exp[-\tfrac{1}{2}\langle(\Sigma_j\phi(r_j,t))^2\rangle] \tag{10}$$

the mean and mean square intensities can, with a little manipulation, be expressed in the form (Jakeman 1974)

$$\langle I(\boldsymbol{r},t)\rangle = \pi^2 W_0^2 E_0^2 \int_0^{\infty} r\,\mathrm{d}r\,\mathrm{J}_0(kr\sin\theta)\exp[-\overline{\phi^2}(1-\rho(r))-r^2/2W_0^2] \tag{11}$$

$$\begin{aligned}\langle I^2(\boldsymbol{r},t)\rangle = \pi W_0^2 E_0^4 \int_{-\infty}^{+\infty} & \mathrm{d}^2r'\,\mathrm{d}^2r''\,\mathrm{d}^2r''' \exp[2\mathrm{i}kr''\sin\theta\cos\eta-(r'^2+r''^2+r'''^2)/W_0^2] \\ & \times\exp\{-\overline{\phi^2}[2-\rho(\boldsymbol{r}''+\boldsymbol{r}''')-\rho(\boldsymbol{r}''-\boldsymbol{r}''')-\rho(\boldsymbol{r}'+\boldsymbol{r}'')-\rho(\boldsymbol{r}'-\boldsymbol{r}'') \\ & +\rho(\boldsymbol{r}'+\boldsymbol{r}''')+\rho(\boldsymbol{r}'-\boldsymbol{r}''')]\}\end{aligned} \tag{12}$$

where η is the polar integration angle corresponding to r'', J_0 is the zeroth-order Bessel function, $\overline{\phi^2}$ is the mean square phase deviation and

$$\rho(r) = \langle\phi(0,t)\phi(\boldsymbol{r},t)\rangle/\overline{\phi^2} \tag{13}$$

is the normalized phase correlation function which we shall take to be translationally invariant and independent of time, corresponding to statistically stationary phase fluctuations. The integrals in equations (11) and (12) can be evaluated using the approximation

$$\exp\overline{\phi^2}\rho(r) \simeq 1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}r^2/\xi^2) \tag{14}$$

which is discussed in Jakeman (1974) and has been used independently by Berry (1973). It is valid if

$$\overline{\phi^2} \gg 1 \tag{15}$$

and if the phase correlation function $\rho(r)$ can be expanded about the origin in terms of a 'correlation' length ξ as follows (Marathay *et al* 1970)

$$\rho(r) = 1-r^2/\xi^2+\ldots. \tag{16}$$

(Note that the presence of a linear term in this expansion would imply discontinuities in $\phi(\boldsymbol{r}, t)$.) Substituting (14) into (11) leads, after integration, to

$$\langle I\rangle = \pi^2 W_0^2 E_0^2 \left\{ W_0^2\,\mathrm{e}^{-\overline{\phi^2}} \exp(-\tfrac{1}{2}k^2 W_0^2 \sin^2\theta) + \frac{1-\mathrm{e}^{-\overline{\phi^2}}}{(1/W_0^2)+(2\overline{\phi^2}/\xi^2)} \exp\left[-\tfrac{1}{2}k^2\left(\frac{\sin^2\theta}{(1/W_0^2)+(2\overline{\phi^2}/\xi^2)}\right)\right]\right\}. \tag{17}$$

The terms in $\exp(-\overline{\phi^2})$ may be neglected for large $\overline{\phi^2}$, as may the $1/W_0^2$ factors, provided that $W_0 \gtrsim \xi$. Thus (17) finally reduces to

$$\langle I\rangle = \frac{\pi^2 W_0^2 \xi^2 E_0^2}{2\overline{\phi^2}} \exp(-k^2\xi^2 \sin^2\theta/4\overline{\phi^2}). \tag{18}$$

Evaluation of the second moment is not so straightforward. Substitution of the approximation (14) into the last exponential factor in (12) leads to the sum of sixteen terms each of which is itself the ratio of two factors. Many of these terms give contributions of order $\exp(-\overline{\phi^2})$ however, and may be neglected. The significant terms are retained in the approximation

$$\exp\{-\overline{\phi^2}[2-\rho(r''+r''')-\rho(r''-r''')-\ldots]\} \simeq \frac{\exp[-\overline{\phi^2}(|\boldsymbol{r}''+\boldsymbol{r}'''|^2+|\boldsymbol{r}''-\boldsymbol{r}'''|^2)/\xi^2]\{2+\mathrm{e}^{2\overline{\phi^2}}\exp[-\overline{\phi^2}(|\boldsymbol{r}'+\boldsymbol{r}''|^2+|\boldsymbol{r}'-\boldsymbol{r}''|^2)]\}}{[1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}|\boldsymbol{r}'-\boldsymbol{r}'''|^2/\xi^2)][1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}|\boldsymbol{r}'+\boldsymbol{r}''|^2/\xi^2)]}. \tag{19}$$

These terms correspond to regions in the three-dimensional vector space $(\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''')$ in which the second factor in the exponent on the left-hand side is small. These are the only regions which contribute to the integral (12) when $\overline{\phi^2} \gg 1$. Thus the first term on the right-hand side of (19) corresponds to the two regions $r'' \sim 0$, $r' \sim 0$ and $r'' \sim 0$, $r''' \sim 0$ (written out separately in equation (A.1)) whilst the second term corresponds to the overlap region $r'' \sim r' \sim r''' \sim 0$. The integrals are evaluated in the appendix and it is shown that after normalization with the mean intensity (18) the second moment may be written (for $W_0 \gtrsim \xi$)

$$\frac{\langle I^2(\boldsymbol{r}, t)\rangle}{\langle I(\boldsymbol{r}, t)\rangle^2} = 2\exp(-\xi^2/W_0^2) + \frac{W_0^2\overline{\phi^2}}{\xi^2}[1-\exp(-\xi^2/2W_0^2)]^2 \exp\left(\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{20}$$

which reduces to the published result (Jakeman and Pusey 1973a, b)

$$\frac{\langle I^2(\boldsymbol{r}, t)\rangle}{\langle I(\boldsymbol{r}, t)\rangle^2} = 2\left(1-\frac{\xi^2}{W_0^2}\right) + \frac{\xi^2\overline{\phi^2}}{4W_0^2}\exp\left(\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{21}$$

in the commonly occurring situation when ξ^2/W_0^2 is small.

2.2. Spatial coherence

From equation (8) the first-order spatial coherence function can be reduced to the form

$$|\langle \mathscr{E}^+(\boldsymbol{r}, t)\mathscr{E}^-(\boldsymbol{r}', t)\rangle| = \pi^2 E_0^2 \int_0^\infty \int_0^\infty r_1\,\mathrm{d}r_1\, r_2\,\mathrm{d}r_2\, \mathrm{J}_0(\tfrac{1}{2}k\boldsymbol{r}_1 \cdot \boldsymbol{u})\mathrm{J}_0(\tfrac{1}{2}k\boldsymbol{r}_2 \cdot \boldsymbol{v}) \times \exp[-\overline{\phi^2}(1-\rho(r_1))]\exp[-(r_1^2+r_2^2)/2W_0^2] \tag{22}$$

where

$$\mathbf{u} = \frac{\mathbf{R}}{r} + \frac{\mathbf{R}'}{r'}, \qquad \mathbf{v} = \frac{\mathbf{R}}{r} - \frac{\mathbf{R}'}{r'}. \tag{23}$$

The integrals may be evaluated within the limits of the approximations (14) and (15) to give ($W_0 \gtrsim \xi$)

$$\frac{|\langle \mathscr{E}^+(\mathbf{r}, t)\mathscr{E}^-(\mathbf{r}', t)\rangle|}{(\langle I(\mathbf{r}, t)\rangle \langle I(\mathbf{r}', t)\rangle)^{1/2}} = \exp\left(\frac{-k^2 W_0^2 v^2}{8}\right). \tag{24}$$

the second-order spatial coherence function is given by the formula

$$\langle I(\mathbf{r}, t) I(\mathbf{r}', t)\rangle = \pi^2 W_0^2 E_0^4 \int_{-\infty}^{+\infty} \mathrm{d}^2 r_1\, d^2 r_2\, \mathrm{d}^2 r_3 \exp[\mathrm{i}k(\mathbf{r}_3 \cdot \mathbf{v} - \mathbf{r}_2 \cdot \mathbf{u})]$$
$$\times \exp[-(r_1^2 + r_2^2 + r_3^2)/W_0^2] \exp[-\overline{\phi^2}(2 - \rho(\mathbf{r}_2 + \mathbf{r}_3) - \rho(\mathbf{r}_2 - \mathbf{r}_3) - \ldots)] \tag{25}$$

by analogy with (12). It is shown in the appendix that within the limits of the approximations outlined in § 2.1 the integral may be performed to obtain the result

$$\frac{\langle I(\mathbf{r}, t) I(\mathbf{r}', t)\rangle}{\langle I(\mathbf{r}, t)\rangle \langle I(\mathbf{r}', t)\rangle} = \left(1 - \frac{\xi^2}{W_0^2}\right)[1 + \exp(-k^2 W_0^2 v^2/4)] + \frac{\xi^2 \overline{\phi^2}}{W_0^2}\left(\frac{\mathrm{J}_1(\frac{1}{2}k\xi v)}{\frac{1}{2}k\xi v}\right)^2$$
$$\times \exp\left(\frac{k^2 \xi^2}{16\overline{\phi^2}}(u^2 + 2v^2)\right) \tag{26}$$

provided that ξ^2/W_0^2 is small.

2.3. *Temporal coherence*

Using the expression (8) for the field, the first-order temporal correlation function of electromagnetic radiation scattered by a random phase screen may be expressed in the form

$$|\langle \mathscr{E}^+(\mathbf{r}, t)\mathscr{E}^-(\mathbf{r}, t+\tau)\rangle| = \pi^2 E_0^2 W_0^2 \int_0^\infty r'\,\mathrm{d}r'\, \mathrm{J}_0(kr' \sin\theta) \exp(-r'^2/2W_0^2)$$
$$\times \exp[-\overline{\phi^2}(1 - c(r', \tau))] \tag{27}$$

where

$$c(r, t) = \langle \phi(0, t)\phi(r, t+\tau)\rangle / \overline{\phi^2}. \tag{28}$$

In order to proceed further we shall assume that the spectrum of phase fluctuations is 'cross-spectrally pure':

$$c(r, \tau) = \rho(r)\sigma(\tau) \tag{29}$$

where $\rho(r)$ is defined by (13) and

$$\sigma(\tau) = \langle \phi(\mathbf{r}, 0)\phi(\mathbf{r}, \tau)\rangle / \overline{\phi^2}. \tag{30}$$

If $\sigma(\tau)$ is close to unity then the main contribution to the integral in equation (27) comes from the region $r' \sim 0$ and using the expansion (16) for ρ this may be evaluated to give, after normalization with the mean intensity (18):

$$\frac{|\langle \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r},t+\tau)\rangle|}{\langle I(\boldsymbol{r},t)\rangle} = \frac{\exp[\overline{\phi^2}(\sigma(\tau)-1)]}{\sigma(\tau)} \exp\left[-\frac{k^2\xi^2}{4\phi^2}\left(\frac{1}{\sigma(\tau)}-1\right)\sin^2\theta\right]. \tag{31}$$

This is finite even in the limit $\sigma(\tau) \to 0$ ($\tau \to \infty$) because of the second exponential factor. However, since $\overline{\phi^2} \gg 1$ the first exponential factor decreases rapidly to zero as $\sigma(\tau)$ deviates from unity. This feature justifies the assumption $\sigma(\tau) \sim 1$ made in order to obtain (31). It will often be a good approximation to write

$$\frac{|\langle \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r},t+\tau)\rangle|}{\langle I(\boldsymbol{r},t)\rangle} = \exp[\overline{\phi^2}(\sigma(\tau)-1)]. \tag{32}$$

The temporal intensity correlation function is given by

$$\begin{aligned}\langle I(\boldsymbol{r},t)I(\boldsymbol{r},t+\tau)\rangle \\ &= \pi W_0^2 E_0^4 \int_{-\infty}^{+\infty} \mathrm{d}^2r'\,\mathrm{d}^2r''\,\mathrm{d}^2r'''\,\exp[2ikr''\sin\theta\cos\eta-(r'^2+r''^2+r'''^2)/W_0^2] \\ &\quad\times\exp\{-\overline{\phi^2}[2-\rho(\boldsymbol{r}''+\boldsymbol{r}''')-\rho(\boldsymbol{r}''-\boldsymbol{r}''') \\ &\quad -\sigma(\tau)(\rho(\boldsymbol{r}'+\boldsymbol{r}'')+\rho(\boldsymbol{r}'-\boldsymbol{r}'')-\rho(\boldsymbol{r}'+\boldsymbol{r}''')-\rho(\boldsymbol{r}'-\boldsymbol{r}'''))]\}\end{aligned} \tag{33}$$

where we have made use of the factorization property (29). At this point we shall assume that $\overline{\phi^2}\sigma(\tau) \gg 1$. Within the limits of the approximations discussed earlier the intensity correlation function turns out to be small outside this region. Thus we can use the approximation

$$\exp(\overline{\phi^2}\rho(r)\sigma(\tau)) \simeq 1+[\exp(\overline{\phi^2}\sigma(\tau))-1]\exp(-\overline{\phi^2}\sigma(\tau)r^2/\xi^2). \tag{34}$$

We need consider only terms analogous to the right-hand side of (19). The first of these gives the usual Gaussian contribution but the second is difficult to evaluate analytically. In the appendix it is shown that results may be obtained in two situations (when ξ^2/W_0^2 is small):

$$\begin{aligned}\frac{\langle I(\boldsymbol{r},t)I(\boldsymbol{r},t+\tau)\rangle}{\langle I(\boldsymbol{r},t)\rangle^2} \\ &= \left(1-\frac{\xi^2}{W_0^2}\right)\left(1+\frac{|\langle \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r},t+\tau)\rangle|^2}{\langle I(\boldsymbol{r},t)\rangle^2}\right)+\frac{\xi^2\overline{\phi^2}}{2W_0^2(1+\sigma(\tau))} \\ &\quad\times\exp\left(\frac{k^2\xi^2\sigma(\tau)\sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right) \qquad \text{for } \sigma(\tau)\overline{\phi^2} \gg 1, \qquad \sigma(\tau) \sim 1\end{aligned} \tag{35}$$

$$\begin{aligned}&= \left(1-\frac{\xi^2}{W_0^2}\right)\left(1+\frac{|\langle \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r},t+\tau)\rangle|^2}{\langle I(\boldsymbol{r},t)\rangle^2}\right)+\frac{\xi^2}{W_0^2(1+\sigma(\tau))} \\ &\quad\times\exp\left(\frac{k^2\xi^2\sigma(\tau)\sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right) \qquad \text{for } \sigma(\tau)\overline{\phi^2} \gg 1, \qquad \sigma(\tau) \ll 1.\end{aligned} \tag{36}$$

These results are somewhat less satisfactory than those obtained in the previous subsections because of the restricted range of times over which they are valid. Calculation of higher-order statistical properties of the field is also difficult using the approach described above, and it is appropriate at this point to consider an alternative method of solution.

3. Micro-area approach

In this method we imagine the scattering region to be a uniformly illuminated disc, but neglect the associated 'aperture' diffraction effects. The emerging wavefront will be divided into a finite number N of micro-areas or facets over which the phase can be regarded as changing in a coherent fashion and it will be assumed that these regions give statistically independent contributions to the far field. This approach has been used with some success for Gaussian fields (ie $N \to \infty$) in connection with light scattering from the sea (see for example Cox and Monk 1954) and diffuse surfaces generally (Enloe 1967, Estes *et al* 1971) but here we shall consider the non-Gaussian situation when N is finite. Our starting point is the expansion

$$\mathscr{E}^{+}(\boldsymbol{r}, t) = \mathrm{e}^{-\mathrm{i}\omega t} \sum_{j=1}^{N} a_j(\boldsymbol{r}, t) \exp(\mathrm{i}\psi_j) \tag{37}$$

for the electromagnetic field at the detector. The ψ_j are independent random phases whilst the diffraction factor a_j for the jth micro-area is given by

$$a_j^2(\boldsymbol{r}, t) = E_0^2 \int_{\mathscr{R}} \int_{\mathscr{R}} \exp \mathrm{i}(k|\boldsymbol{r}' - \boldsymbol{r}''| \sin\theta \cos\eta + \phi_j(\boldsymbol{r}', t) - \phi_j(\boldsymbol{r}'', t))\, \mathrm{d}^2 r'\, \mathrm{d}^2 r'' \tag{38}$$

where η is the angular variable corresponding to the polar vector $\boldsymbol{r}' - \boldsymbol{r}''$ and $\mathscr{R}$ is a region of coherently changing phase or 'facet' on the wavefront emerging from the phase screen. In order to evaluate the moments of the a_j we must still assume some form for the statistical properties of the ϕ_j. We shall make the approximation that ϕ varies linearly over the regions $\mathscr{R}$ which are of dimensions ξ:

$$\phi = \boldsymbol{r} \,.\, \boldsymbol{m}/\xi \tag{39}$$

where the slope is (two-dimensionally) Gaussian distributed

$$p(\boldsymbol{m}) = \frac{\exp(-m^2/4\overline{\phi^2})}{4\pi\overline{\phi^2}}. \tag{40}$$

These assumptions are equivalent to the joint-Gaussian model adopted in § 2 (Jakeman 1974). In order to evaluate temporal correlation functions we shall make the related assumption

$$p(\boldsymbol{m}, \boldsymbol{n}) = \frac{\xi^4}{(4\pi\overline{\phi^2})^2(1-\sigma^2(\tau))} \exp\left(-\frac{\xi^2}{4\phi^2}\frac{m^2 + n^2 - 2\boldsymbol{m}\,.\,\boldsymbol{n}\sigma(\tau)}{1-\sigma^2(\tau)}\right). \tag{41}$$

It is not difficult to demonstrate that the joint-Gaussian model plus the factorization property (29) lead to the same result for the field correlation function $\langle \exp \mathrm{i}\, \Sigma_j (\phi(r_j', t_j) - \phi(r_j'', t_j) \rangle$ as (39) and (41) so that these assumptions are equivalent.

3.1. Statistics

Assuming that the a_j are all described by the same probability distribution the mean and mean square intensities may be written from (37) in the form

$$\langle I \rangle = N\langle a^2 \rangle \tag{42}$$

$$\frac{\langle I^2 \rangle}{\langle I \rangle^2} = 2\left(1 - \frac{1}{N}\right) + \frac{1}{N}\frac{\langle a^4 \rangle}{\langle a^2 \rangle^2}. \tag{43}$$

Here we have taken advantage of the statistical independence of the phase factors in (37) and of the a_j. In order to estimate the size of the diffraction factors we shall confine the region of integration in (38) to a disc of radius $\xi/\sqrt{2}$. This is consistent with the expansion (16) for the phase coherence function although it is not, of course, possible in practice to decompose the scattering region into a set of non-overlapping discs. Substitution from (39) into (38) leads after integration to

$$a^2 = \left(\sqrt{(2)}\pi\xi E_0 \frac{J_1(\xi|\boldsymbol{k}\sin\theta + \boldsymbol{m}|/\sqrt{2})}{|\boldsymbol{k}\sin\theta + \boldsymbol{m}|}\right)^2. \tag{44}$$

In calculating the averages of powers of this quantity using the distribution (40) we use the approximation (Watson 1944, p 421)

$$\frac{J_1(x)}{x} = \tfrac{1}{2}\,e^{-x^2/8} \tag{45}$$

since the major contribution to the integrals comes from the region $x \sim 0$. This gives, for $\overline{\phi^2} \gg 1$ and $n \geqslant 1$ (Jakeman and Pusey 1973a, Jakeman 1974)

$$\langle a^{2n}\rangle = \left(\frac{\pi\xi^2 E_0}{2}\right)^{2n} \frac{2}{n\overline{\phi^2}} \exp\left(\frac{-k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{46}$$

so that (42) and (43) may be expressed in the form

$$\langle I(\boldsymbol{r},t)\rangle = \frac{N\pi^2\xi^4E_0^2}{2\overline{\phi^2}} \exp\left(\frac{-k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{47}$$

$$\frac{\langle I^2(\boldsymbol{r},t)\rangle}{\langle I(\boldsymbol{r},t)\rangle^2} = 2\left(1-\frac{1}{N}\right) + \frac{\overline{\phi^2}}{4N}\exp\left(\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right). \tag{48}$$

These results are identical with (18) and (21) of § 2.1 when we set $N = W_0^2/\xi^2$. This is a reasonable identification since $W_0/\sqrt{2}$ is the width of the field amplitude profile of the phase screen and we have taken phase correlation regions of radius $\xi/\sqrt{2}$.

3.2. Spatial coherence functions

The first-order spatial coherence function can be evaluated as described in § 2.2 and we consider here only the intensity correlation function. This may be written from (37) in the form

$$\frac{\langle I(\boldsymbol{r},t)I(\boldsymbol{r}',t)\rangle}{\langle I(\boldsymbol{r},t)\rangle\langle I(\boldsymbol{r}',t)\rangle} = \left(1-\frac{1}{N}\right)\left(1+\frac{|\langle \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r}',t)\rangle|^2}{\langle I(\boldsymbol{r},t)\rangle\langle I(\boldsymbol{r}',t)\rangle}\right) + \frac{1}{N}\frac{\langle a^2(\boldsymbol{r},t)a^2(\boldsymbol{r}',t)\rangle}{\langle a^2(\boldsymbol{r},t)\rangle\langle a^2(\boldsymbol{r}',t)\rangle}. \tag{49}$$

The last term may be evaluated with the help of (40), (44) and (45) to give

$$\frac{\langle I(\boldsymbol{r},t)I(\boldsymbol{r}',t)\rangle}{\langle I(\boldsymbol{r},t)\rangle\langle I(\boldsymbol{r}',t)\rangle}$$
$$= \left(1-\frac{1}{N}\right)\left(1+\frac{|\langle \mathscr{E}^+(\boldsymbol{r},t)\mathscr{E}^-(\boldsymbol{r}',t)\rangle|^2}{\langle I(\boldsymbol{r},t)\rangle\langle I(\boldsymbol{r}',t)\rangle}\right) + \frac{\overline{\phi^2}}{4N}\exp(-k^2\xi^2v^2/16)$$
$$\times \exp\frac{k^2\xi^2}{16\overline{\phi^2}}(u^2+2v^2), \tag{50}$$

where u and v were defined in § 2.2. This formula differs slightly from that derived using the analytical approach (equation (26)).

3.3. Temporal coherence functions

The first-order temporal correlation function can again most easily be calculated by the analytical approach given in § 2 and we shall evaluate only the intensity correlation function here. This takes the form

$$\frac{\langle I(\mathbf{r},t)I(\mathbf{r},t+\tau)\rangle}{\langle I(\mathbf{r},t)\rangle^2} = \left(1-\frac{1}{N}\right)\left(1+\frac{|\langle \mathscr{E}^+(\mathbf{r},t)\mathscr{E}^-(\mathbf{r},t+\tau)\rangle|^2}{\langle I(\mathbf{r},t)\rangle^2}\right)+\frac{1}{N}\frac{\langle a^2(\mathbf{r},t)a^2(\mathbf{r},t+\tau)\rangle}{\langle a^2(\mathbf{r},t)\rangle^2}. \tag{51}$$

The last term may be evaluated from the definition (38) using (39), (41) and (45). The final result is

$$\frac{\langle I(\mathbf{r},t)I(\mathbf{r},t+\tau)\rangle}{\langle I(\mathbf{r},t)\rangle^2} = \left(1-\frac{1}{N}\right)\left(1+\frac{|\langle \mathscr{E}^+(\mathbf{r},t)\mathscr{E}^-(\mathbf{r},t+\tau)\rangle|^2}{\langle I(\mathbf{r},t)\rangle^2}\right) + \frac{\exp[k^2\xi^2\sigma(\tau)\sin^2\theta/2\overline{\phi^2}(1+\sigma(\tau))]}{N\{1-[\overline{\phi^2}\sigma(\tau)/(2+\overline{\phi^2})]^2\}}. \tag{52}$$

When N is set equal to W_0^2/ξ^2 this formula reproduces the analytical results (35) and (36) in the appropriate limits ($\sigma \sim 1$ and $\sigma \ll 1$ respectively). Moreover, (52) behaves correctly in the limit $\tau \to \infty$ ($\sigma \to 0$) when it reduces to unity.

3.4. Two-time, two-point correlation function

It is not difficult using the micro-area approach to derive a formula which combines the statistical, spatial and temporal coherence properties contained in equations (48), (50) and (52). Thus the most general two-time, two-point correlation function of the intensity of radiation scattered by a Gaussian random phase screen may, according to this method, be expressed in the form

$$\frac{\langle I(\mathbf{r},t)I(\mathbf{r}',t+\tau)\rangle}{\langle I(\mathbf{r},t)\rangle\langle I(\mathbf{r}',t)\rangle} = \left(1-\frac{1}{N}\right)\left(1+\frac{|\langle \mathscr{E}^+(\mathbf{r},t)\mathscr{E}^-(\mathbf{r}',t+\tau)\rangle|^2}{\langle I(\mathbf{r},t)\rangle\langle I(\mathbf{r}',t)\rangle}\right) + \frac{\exp(-k^2\xi^2v^2/16)\exp[k^2\xi^2(u^2+2v^2)/16\overline{\phi^2}]}{N\{1-[\overline{\phi^2}\sigma(\tau)/(2+\overline{\phi^2})]^2\}}\exp\left(\frac{k^2\xi^2\sigma(\tau)\sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right), \tag{53}$$

where the normalized first-order correlation function is given by

$$\frac{|\langle \mathscr{E}^+(\mathbf{r},t)\mathscr{E}^-(\mathbf{r}',t+\tau)\rangle|}{(\langle I(\mathbf{r},t)\rangle\langle I(\mathbf{r}',t)\rangle)^{1/2}} = \exp\left(-\frac{k^2W_0^2v^2}{8}-\overline{\phi^2}(1-\sigma(\tau))\right), \tag{54}$$

the mean intensity by

$$\langle I(\boldsymbol{r}, t)\rangle \propto \frac{N\xi^4}{\overline{\phi^2}} \exp\left(-\frac{k^2\xi^2 \sin^2\theta}{4\overline{\phi^2}}\right) \tag{55}$$

and u and v by equation (23).

3.5. *Higher-order statistics*

It is possible, using the micro-area approach to calculate higher-order statistical properties of the field. This problem is discussed in previous publications (Jakeman and Pusey 1973a, Jakeman 1974) and we include here a brief summary of the method of solution for completeness but without further comment. The method is based on the observation that (37) is a two-dimensional random walk of variable step length. The distribution of the resultant of such a walk has been investigated by several authors (see for example Rayleigh 1919 and references therein) and the subject has recently received renewed interest in connection with the scattering of light from finite numbers of particles (Pusey *et al* 1974, Schaefer 1974). The formula relevant to the present work is quoted by Watson (1944, p 420)

$$P(I) = \tfrac{1}{2} \int_0^\infty u \mathrm{J}_0(u\sqrt{I}) \prod_{i=1}^{N} \mathrm{J}_0(a_i u)\, \mathrm{d}u. \tag{56}$$

The generating function corresponding to this distribution is a confluent hypergeometric function of N variables (see, for example, Erdelyi 1954, p 385)

$$\langle \exp(-sI)\rangle = \psi_2(1\,;1, 1, \ldots 1\,; -a_1^2 s, -a_2^2 s, \ldots -a_N^2 s) \tag{57}$$

from which the moments may be derived without difficulty. After averaging over the step lengths (diffraction factors) we obtain (Jakeman 1974)

$$\langle I^n\rangle = (n!)^2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_N=0}^{\infty} \prod \langle a^{2m_i}\rangle \left(\prod_i m_i!\right)^{-2} \Bigg|_{\Sigma_i m_i = n} \tag{58}$$

which may be evaluated explicitly using (46). Results for up to the fifth intensity moment, and comparison with experiment are presented in Jakeman and Pusey (1973a) and Jakeman (1974).

4. Discussion and comparison of theoretical results

Before embarking on a detailed analysis of the results derived above, it is useful to express the assumptions and approximations made in § 2 in terms of more familiar physical concepts so as to highlight the similarities between the two methods of approach to the problem which we have described.

We have already pointed out the equivalence between the assumption of joint-Gaussian phase statistics together with the expansion (16) for $\rho(r)$ made in the analytical approach of § 2 and the linearly-faceted model with Gaussian slope distribution adopted in § 3. The choice of joint-Gaussian statistics for the phase is a somewhat arbitrary one, though it is commonly made in the literature both for scattering from rough surfaces (Beckmann and Spizzichino 1963) and for propagation through phase screens (Mercier

1962) either for mathematical convenience or for want of a better model. It is well known, however, that other distributions lead to radically different angular distributions of scattered intensity (Kivelson and Moszkowski 1965, Beckmann 1973).

The assumption $\overline{\phi^2} \gg 1$ has several implications. It means that different parts of the wavefront are out of phase by randomly large amounts equivalent to path differences of the order of a wavelength or more. Thus the approximations (14) and (19) and the discarding of terms like $\exp(-\overline{\phi^2})$ in the analytical approach are equivalent to the assumption of *independently* contributing micro-areas made in § 3. Since $(k\xi)^{-1}$ may be regarded as the angular spread of the diffraction lobe from a single facet, whilst $\sqrt{\overline{\phi^2}}/k\xi$ corresponds to the width of the overall angular distribution of scattered intensity (equation (18)) a second implication of large $\overline{\phi^2}$ is that the facet slope distribution rather than the facet size will govern the scattered intensity distribution. Furthermore $(kW_0)^{-1}$ represents the width of the far-field diffraction pattern of the entire scattering region so that $(\overline{\phi^2}W_0^2/\xi^2)^{1/2}$ is the ratio of the width of the scattered intensity distribution to the 'speckle' size, and will be large provided that W_0 is not much smaller than ξ. Neglecting terms containing higher inverse powers of this quantity in the analytical method is thus seen to be entirely consistent with the neglect of 'aperture' diffraction effects in the micro-area approach.

With the exception of equation (20), we have presented for simplicity in § 2 results valid only in the limit $\xi^2/W_0^2 \ll 1$ (but finite). In fact it is clear from the formulae (20), (21) and (48) for the second intensity moment that the micro-area and analytical methods give identical results only in this limit. This is because we have assumed a Gaussian intensity profile for the incident beam of radiation in the analytical method but a uniformly illuminated scattering region in the micro-area approach. The effect of using a Gaussian intensity profile in this latter picture would be two-fold. Firstly, it would increase the effective number of scatterers. Secondly, it would introduce another statistical fluctuation in the scattering cross section of each facet due to the range of its possible positions relative to the centre of the beam. These fluctuations would tend to be averaged somewhat by the finite size of the facets. The result of the interplay of these effects is difficult to predict, but according to equation (20), a decrease in the non-Gaussian term and an increase in the Gaussian contribution to the second intensity moment is produced when they are taken into account. However, it is quite possible that both the Gaussian and non-Gaussian contributions to the higher-order moments might be increased by using a Gaussian beam profile rather than a uniformly illuminated disc.

A further factor complicating a comparison of the two theoretical approaches is the assumption of constant ξ and N in § 3. This does not lead to differences with the analytical approach however, for the following reason. Examination of the integrands in the expressions which must be evaluated to obtain the formulae of § 2, for example (A.9), indicates that they are generally characterized by relatively sharp cut-off regions which contribute only negligible fractions (of order $1/\overline{\phi^2}$) of the whole integrals. These cut-off regions correspond, in the micro-area picture, to the spread of facet sizes: so the neglect of such a spread and the consequent variations in N is consistent with the analytical treatment. The width of the cut-off regions is in fact closely related to the spatial decay of the electric field correlation function at the phase screen, whilst their shape will be sensitive to the approximation (16) for the phase coherence function. It is difficult to estimate the effect of this approximation but it is probably not a significant factor as far as the second moment is concerned. However, it seems likely that the distribution of facet size and number should be properly taken into account in the micro-area approach when calculating either the higher-order statistical properties of the intensity (which are dominated by fluctuations in a_j, defined by equation (38)) or the spatial coherence properties (which are determined in part by characteristics of the individual micro-areas—see (*b*) below).

From the preceding discussion it appears that the two theoretical treatments described in § 2 and 3 are almost entirely equivalent, at least in the limit $\xi^2/W_0^2 \ll 1$, and it is not surprising that similar results are obtained. The micro-area approach may be thought of as a kind of 'diagram' technique (see for example Zipfel and De Santo 1972) for summing the important contributions to the integrals which must be evaluated in the analytical treatment of § 2. Thus the two important regions of integration referred to in § 2.1 correspond to the beating of intensities from different facets ($r'' \sim 0$; $r' \sim 0$ or $r''' \sim 0$) and to the beating of intensities from the same facet ($r'' \sim r' \sim r''' \sim 0$) and give rise to the Gaussian (first) and non-Gaussian (second) terms in the second-order statistic (21).

We shall now revert to the pattern adopted in the last two sections and discuss separately the statistics, spatial and temporal coherence properties predicted for radiation scattered by a deep random phase screen.

4.1. *Statistics*

The mean intensity (17), (18) or (47) exhibits the broad angular spread and almost total absence of direct beam typical of radiation scattered by a deep random phase screen or very rough surface (see § 5). The main feature of the distribution is that the characteristics of the scatterer enter into the formula only as the ratio $\xi/\sqrt{\overline{\phi^2}}$, ie the inverse RMS slope of the wavefront. Thus measurements of $\langle I \rangle$ cannot be used to determine ξ or $\overline{\phi^2}$ separately. This may be contrasted with the second-order statistic (20), (21) or (48) which depends on the model parameters in a more complicated way. The self-evident departure from the Gaussian value of two diminishes as $\xi^2/W_0^2 \to 0$, but even for small values of this ratio when (21) and (48) are valid this deviation may be considerable owing to the large $\overline{\phi^2}$ factor multiplying the second term. As we shall see in the next section this is intimately related to the normalization of the slope distribution $p(\boldsymbol{m})$. In many situations (21) and (48) will be adequate so that by measuring the angular dependence of the second moment and intercept at $\theta = 0$ both the RMS slope, $\xi/\sqrt{\overline{\phi^2}}$, and the product $\xi\sqrt{\overline{\phi^2}}$ can be determined. This technique has in fact already been used to determine the parameters characterizing the dynamic scattering mode exhibited by thin layers of nematic liquid crystal (Jakeman and Pusey 1973b) and is discussed further in II. The non-Gaussian effect is enhanced at large angles by the exponential factor corresponding to the low probability of finding facets of the wavefront tilted at large angles. The dependence of the second moment on angle is just the inverse of that shown by the mean intensity (18) or (47). This is consistent with the predictions of the simple model given in the introduction.

4.2. *Spatial coherence*

The dependence of the first-order spatial coherence function (24) on the detector separation through the quantity v is that expected from an incoherent source of Gaussian intensity profile situated in the plane of the phase screen, and corresponds to a far-field speckle size or coherence length of angular width $(kW_0)^{-1}$. The usual spatial coherence factor, depending on the source size or illuminated area also appears in the Gaussian (first) term of the second-order coherence function (26). The non-Gaussian term is characterized by the decay length expected from a source of radius ξ, however, and will fall off more slowly if $W_0 > \xi$. Thus our model predicts two coherence lengths: one related to the diffraction pattern of the scattering region and one to the correlation length of the phase fluctuations. The corresponding non-Gaussian (second) term of the result (50) obtained by the micro-area approach displays a slightly different dependence on v unless $\frac{1}{2}kv\xi \ll 1$. The particular behaviour obtained by the micro-area approach is in fact a direct consequence of the approximation (45) which is itself equivalent to assuming a smooth convergence factor $4\exp[-4(r'^2+r''^2)/\xi^2]$ in the integrals (38) rather than a sharp cut-off at $r' = \xi$ and $r'' = \xi$. The Bessel function behaviour is thus to be expected from a model in which the regions of coherent phase are rather well defined. As mentioned earlier, inspection of the integrals in the appendix arising from the approximation (19) of the analytic treatment indicates that it does indeed correspond to such a model.

One important conclusion which must be drawn from the above discussion is that whereas the non-Gaussian contributions to the statistics (and, as we shall see, the spectrum) are relatively insensitive to the detailed nature of the facets and are therefore almost model-independent, the non-Gaussian contribution to the spatial coherence function is closely connected with the diffraction pattern of a single facet and is therefore more intimately related to the detailed structure of the scattering fluctuations. Although the main properties predicted for this function in §§ 2 and 3 are therefore likely to be correct, its detailed behaviour may well not be right for every scattering system.

4.3. *Temporal coherence*

We have already seen that the results of the micro-area approach reduce to the asymptotic formulae obtained by the analytical method in the appropriate limits. The first-order temporal coherence function (31) or (32) decreases rapidly as the delay time τ increases from zero and is therefore only sensitive to the behaviour of $\sigma(\tau)$ for small values of τ. In this region $\sigma(\tau)$ may be expanded as a Taylor series about $\tau = 0$. Thus, for example, a Gaussian phase correlation function $\sigma(\tau)$ of characteristic decay time τ_c gives rise to a Gaussian field correlation function of coherence time $\tau_c/\sqrt{\overline{\phi^2}}$. This is consistent with the introduction of Doppler shifts into the incident radiation corresponding to path changes of order $\sqrt{\overline{\phi^2}}/k$ in the time τ_c of the phase fluctuations. The Gaussian contribution to the second-order correlation function will show a similar time dependence according to (35), (36) and (52) by virtue of the Siegert-type factorization (Siegert 1943) in terms of the field correlation function. The non-Gaussian term in (52) exhibits a more complex behaviour, however. Since $\overline{\phi^2} \gg 1$ the prefactor $\{1-[\overline{\phi^2}\sigma(\tau)/(2+\overline{\phi^2})]^2\}^{-1}$ decreases rapidly from $\overline{\phi^2}/4$ at $\tau = 0$ to $1/(1-\sigma(\tau))$ at small nonzero values of τ. Indeed an expansion of this quantity about $\tau = 0$ shows that it has a Lorentzian shape (negative exponential spectrum) of width $\tau_c/\sqrt{\overline{\phi^2}}$ when $\sigma(\tau)$ is Gaussian. After the rapid fall of this factor to a value near unity, however, the time dependence of the non-Gaussian term is controlled by the angle-dependent exponential factor. When the exponent is small, ie for sufficiently large τ, only the first terms of an expansion in powers of the argument need be retained and (52) reduces to

$$\frac{\langle I(\boldsymbol{r},t)I(\boldsymbol{r},t+\tau)\rangle}{\langle I(\boldsymbol{r},t)\rangle^2} = 1+\frac{k^2\xi^2\sigma(\tau)\sin^2\theta}{2N\overline{\phi^2}}. \tag{59}$$

Thus the long tail of the correlation function is a direct measure of the phase fluctuations. In terms of the simple facet model described in the introduction, the origin of the two time scales may be understood as follows. The short correlation time is the duration of the sweep of diffraction lobes of angular width $(k\xi)^{-1}$ from individual facets on the emerging wavefront across the (point) detector. The overall angular sweep (ie the angular spread of the scattered intensity) is of the order of $\sqrt{\overline{\phi^2}}/k\xi$ which takes place in a time τ_c. Thus each facet illuminates the detector on average for a characteristic time

$$\tau_c(k\xi)^{-1}k\xi/\sqrt{\overline{\phi^2}} = \tau_c/\sqrt{\overline{\phi^2}}.$$

The long decay time, on the other hand, is related to the likelihood of a facet returning to its previous orientation, ie to the joint probability of finding a particular slope at time t and time $t+\tau$. It is therefore not surprising that it is of the same order of magnitude as the phase correlation time τ_c itself, as indicated by (59) above.

5. Application to rough surfaces

In this section we shall consider the transformations necessary for our results to be applicable to the scattering of electromagnetic radiation from rough surfaces. A good deal of literature exists on this subject but as far as we are aware previous work has concentrated exclusively on the Gaussian regime when the spatial scale of the surface structure is small compared to the total scattering area. A full discussion of the complications of finite surface reflectance and depolarization effects may be found in the book by Beckmann and Spizzichino (1963) and we shall confine ourselves here to the simple case of a perfectly conducting surface.

We have remarked on several occasions that a rough surface behaves as a phase screen. The details of the equivalence may be deduced with the help of figure 3 which

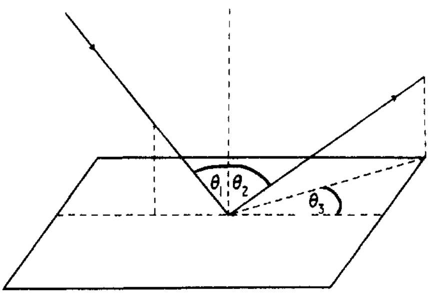

Figure 3. General geometry for scattering from a rough surface.

shows the geometry of a general surface scattering configuration. For a perfectly conducting surface equation (8) for the field in the Fraunhofer region is replaced by (Beckmann and Spizzichino 1963, chap 2)

$$\mathscr{E}^+(\boldsymbol{r},t) = F_0 \exp[\mathrm{i}(kr-\omega t)] \int_{-\infty}^{+\infty} \mathrm{d}^2\boldsymbol{r}' \exp[\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2).(\boldsymbol{r}'+z\hat{\boldsymbol{z}})]A(\boldsymbol{r}') \qquad (60)$$

where

$$F_0 \propto \frac{1+\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\theta_3}{\cos\theta_1+\cos\theta_2} \qquad (61)$$

and $A(\boldsymbol{r}')$ defines the variation of intensity within the illuminated region. $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the incident and scattered wavevectors of the radiation, $\boldsymbol{r}'$ is a polar vector measured in the plane of the surface and z the local surface height measured from this median plane. Since

$$\boldsymbol{k}_1-\boldsymbol{k}_2 = \boldsymbol{k}[(\sin\theta_1-\sin\theta_2\cos\theta_3)\hat{\boldsymbol{x}}-\sin\theta_2\sin\theta_3\hat{\boldsymbol{y}}-(\cos\theta_1+\cos\theta_2)\hat{\boldsymbol{z}}] \tag{62}$$

the usual scattering vector term appearing in equation (8) of § 2 which may be written $\exp \mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}'\sin\theta$ is replaced by

$$\exp\{\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{r}'[(\sin\theta_1-\sin\theta_2\cos\theta_3)^2+\sin^2\theta_2\sin^2\theta_3]^{1/2}\}. \tag{63}$$

The second contribution to the phase shift in (60), ie

$$(\boldsymbol{k}_1-\boldsymbol{k}_2)\,.\,\hat{\boldsymbol{z}}z = kz(\cos\theta_1+\cos\theta_2) \tag{64}$$

depends on the surface profile and replaces the $\exp(\mathrm{i}\phi(\boldsymbol{r}', t))$ factor in equation (8). Thus in order to derive formulae applicable to the scattering of electromagnetic radiation from rough surfaces from our earlier results the following transformations are necessary:

$$\sin\theta \rightarrow [(\sin\theta_1-\sin\theta_2\cos\theta_3)^2+\sin^2\theta_2\sin^2\theta_3]^{1/2} \tag{65}$$

$$\overline{\phi^2} \rightarrow k^2\overline{z^2}(\cos\theta_1+\cos\theta_2)^2 \tag{66}$$

$$E_0 \rightarrow F_0 \tag{67}$$

where $\overline{z^2}$ is the mean square height deviation of the surface. The deep phase screen limit $\overline{\phi^2} \gg 1$ is thus seen to be equivalent to the very rough surface condition $\overline{z^2}^{1/2} \gg \lambda/2\pi$. A particularly simple result for the statistics is obtained at normal incidence when $\theta_1 = \theta_3 = 0$ and when $A(\boldsymbol{r}')$ is Gaussian as in equation (8):

$$\langle I\rangle \propto \sec^4\theta/2 \exp\left(-\frac{\xi^2\tan^2\theta/2}{4\overline{z^2}}\right) \tag{68}$$

$$\frac{\langle I^2\rangle}{\langle I\rangle^2} = 2\left(1-\frac{\xi^2}{W_0^2}\right)+\frac{k^2\xi^2\overline{z^2}\cos^4\theta/2}{W_0^2}\exp\left(\frac{\xi^2\tan^2\theta/2}{4\overline{z^2}}\right). \tag{69}$$

The first result is already to be found in the literature (see for example Beckmann and Spizzichino 1963, chap 5) but the deviation from Gaussian statistics implicit in equation (69) has not been studied previously as far as we are aware. For a given surface roughness this deviation increases as the wavevector of the incident radiation increases so that non-Gaussian fluctuations can be obtained by scattering sufficiently short wave radiation from any moving surface even when very many facets are illuminated. Moreover, as we have seen, these fluctuations are entirely an intensity effect so that broadband radiation satisfying minimal conditions of coherence could in principle be used to obtain the distribution of surface slopes *and* correlation length ξ from measurements of the second-order statistic (69).

Although relations (68) and (69) are only valid for a surface whose height is Gaussian distributed with respect to a median reference plane, the simple geometrical optics approach described in the introduction strongly suggests that it might be possible to derive more generally applicable formulae. Indeed Kivelson and Moszkowski (1965) have shown rigorously (within the limits of diffraction theory) that the back-scattering cross section of a very rough, perfectly conducting surface is a function only of the (arbitrary) surface slope distribution—a result which is implicit in equation (5). Extension of the method used by these authors to the calculation of *second*-order statistical properties of the intensity of radiation scattered by a surface of arbitrary slope distribution would appear to be a formidable task. However, a generalization of the micro-area approach of § 3 seems more feasible and may well provide an adequate basis for the development of a new technique for the measurement of surface roughness. (Current methods of characterization and measurement of micron-scale surface roughness are reviewed by Spragg and Whitehouse (1971) and Sprague (1972).)

The transformations (65)–(67) can also be applied to the spectral properties (32) and (52) and the resulting formulae might find application in studies of intrinsically fluctuating surfaces such as the sea. It is interesting in this context, that over twenty years ago Goldstein (1953) suggested that the effective number of scatterers might be determined by studying deviation from Gaussian statistics. The 'cross-spectral purity' condition (29) does not hold in general, however, for rigid surfaces undergoing simple translational motions such as moving ground glass, and the temporal correlation properties of this type of system have yet to be investigated in the non-Gaussian regime.

Transformation of the spatial coherence properties (24) and (26) is more difficult unless the two detectors are sufficiently close for the RMS phase shift introduced by the surface to be the same in both directions θ and θ'. θ_2 may then be taken as the mean of these angles in (66) whilst the transformation (65) may be applied without difficulty to u^2 and v^2 when these are expressed in the form

$$\sin^2\theta + \sin^2\theta' \pm 2 \sin\theta \sin\theta' \cos\psi$$

where ψ is the angle between $\boldsymbol{R}$ and $\boldsymbol{R}'$.

6. Conclusions

We have shown that when electromagnetic radiation is scattered by a deep random phase screen then fluctuations in the scattered radiation, which arise when the scattering region is small, may be related in a quantitative way to elementary properties of the scatterer. In particular the mean square intensity and the temporal and spatial coherence functions of the scattered radiation may be expressed in a form in which departures from Gaussian statistics are explicit functions of these properties. Gaussian (field) statistics are to be expected, by virtue of the central limit theorem, when many scatterers contribute to the far-field intensity. However our results indicate that non-Gaussian effects may be important even when the effective number of scattering centres is large. The central limit theorem must therefore be used with some care when predicting statistical properties of scattered radiation.

We have demonstrated the mathematical equivalence of a Gaussian random phase screen model and an approach in which the wavefront of the scattered radiation is assumed to be a linearly faceted structure with a Gaussian slope distribution.

Finally, we have indicated how our formulae may be modified in order that they may be applied to scattering from very rough surfaces and indicated that certain of the results might find application in the measurement of surface roughness. Much remains to be done in this field however and we have not attempted a detailed investigation of the technique here. This will form the subject of a future publication.

Acknowledgments

We should like to thank Dr J G McWhirter for his help in the evaluation of certain of the integrals appearing in the appendix of this work.

Appendix

A.1. Moments

We shall first calculate the mean square intensity (20) using the approximation (19). Two integrals must be evaluated:

$$I_1 = \pi W_0^2 E_0^4 \int_{-\infty}^{+\infty} \mathrm{d}^2r'\,\mathrm{d}^2r''\,\mathrm{d}^2r'''\exp[2ikr''\sin\theta\cos\eta-(r'^2+r''^2+r'''^2)/W_0^2]$$
$$\times\frac{\exp[-\overline{\phi^2}(|\boldsymbol{r}''+\boldsymbol{r}'''|^2+|\boldsymbol{r}''-\boldsymbol{r}'''|^2)/\xi^2]+\exp[-\overline{\phi^2}(|\boldsymbol{r}''+\boldsymbol{r}'|^2+|\boldsymbol{r}''-\boldsymbol{r}'|^2)/\xi^2]}{[1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}|\boldsymbol{r}'-\boldsymbol{r}'''|^2/\xi^2)][1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}|\boldsymbol{r}'+\boldsymbol{r}'''|^2/\xi^2)]} \tag{A.1}$$

and

$$I_2 = \pi W_0^2 E_0^4\,\mathrm{e}^{2\overline{\phi^2}} \int_{-\infty}^{+\infty} \mathrm{d}^2r'\,\mathrm{d}^2r''\,\mathrm{d}^2r'''\exp[2ikr''\sin\theta\cos\eta-(r'^2+r''^2+r'''^2)/W_0^2]$$
$$\times\frac{\exp[-\overline{\phi^2}(|\boldsymbol{r}''+\boldsymbol{r}'''|^2+|\boldsymbol{r}''-\boldsymbol{r}'''|^2+|\boldsymbol{r}'+\boldsymbol{r}''|^2+|\boldsymbol{r}'-\boldsymbol{r}''|^2)/\xi^2]}{[1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}|\boldsymbol{r}'-\boldsymbol{r}'''|^2/\xi^2)][1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}|\boldsymbol{r}'+\boldsymbol{r}'''|^2/\xi^2)]}. \tag{A.2}$$

The r'' integration can be carried out exactly in both cases leading to

$$I_1 = \frac{\pi^2W_0^2\xi^2E_0^4}{2\overline{\phi^2}}\exp\left(-\frac{k^2\xi^2\sin^2\theta}{2\overline{\phi^2}}\right)\int_{-\infty}^{+\infty}\mathrm{d}^2r'\,\mathrm{d}^2r'''$$
$$\times\frac{\exp[-(r'^2+r'''^2)/W_0^2][\exp(-2\overline{\phi^2}r'''^2/\xi^2)+\exp(-2\overline{\phi^2}r'^2/\xi^2)]}{[\ldots][\ldots]} \tag{A.3}$$

$$I_2 = \frac{\pi^2W_0^2\xi^2E_0^4}{4\overline{\phi^2}}\mathrm{e}^{2\overline{\phi^2}}\exp\left(-\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right)\int_{-\infty}^{+\infty}\mathrm{d}^2r'\,\mathrm{d}^2r'''$$
$$\times\frac{\exp\{-(r'^2+r'''^2)[(1/W_0^2)+(2\overline{\phi^2}/\xi^2)]\}}{[\ldots][\ldots]}. \tag{A.4}$$

The kernel of the first integral is symmetric in r' and r''' and is dominated by the convergence factors $\exp(-2\overline{\phi^2}r'''^2/\xi^2)$ and $\exp(-2\overline{\phi^2}r'^2/\xi^2)$. The significant contributions to I_1 are retained by using an approximation of the type:

$$\lim_{\overline{\phi^2}\to\infty}\frac{2\overline{\phi^2}}{\pi\xi^2}\exp(-2\overline{\phi^2}r'''^2/\xi^2) = \delta^2(\boldsymbol{r}''') \tag{A.5}$$

for both the r''' and r' terms. This leads to

$$I_1 = \frac{\pi^4W_0^2\xi^4E_0^4}{2\overline{\phi^2}^2}\exp\left(-\frac{k^2\xi^2\sin^2\theta}{2\overline{\phi^2}}\right)A \tag{A.6}$$

where

$$A = \int_0^\infty\frac{\mathrm{d}x\,\mathrm{e}^{-x/W_0^2}}{[1+(\mathrm{e}^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}x/\xi^2]^2}. \tag{A.7}$$

A transformation to sum and difference coordinates in I_2 gives, after performing the angular integrations:

$$I_2 = \frac{\pi^4 W_0^2 \xi^2 E_0^4 \, e^{2\overline{\phi^2}}}{16\overline{\phi^2}} \exp\left(-\frac{k^2\xi^2 \sin^2\theta}{4\overline{\phi^2}}\right) B^2 \tag{A.8}$$

where

$$B = \int_0^\infty \mathrm{d}x \frac{\exp\{-x[(1/2W_0^2)+(\overline{\phi^2}/\xi^2)]\}}{1+(e^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}x/\xi^2)}. \tag{A.9}$$

A and B may be evaluated by dividing the region of integration into two regions in which the denominators may be expanded:

$$A = \sum_{n=0}^{\infty} (n+1)(-1)^n \left(\int_0^\eta \frac{\exp\{x[-1/W_0^2+(n+2)\overline{\phi^2}/\xi^2]\}}{(e^{\overline{\phi^2}}-1)^{n+2}} \mathrm{d}x + \int_\eta^\infty \frac{\exp[-x(1/W_0^2+n\overline{\phi^2}/\xi^2)]}{(e^{\overline{\phi^2}}-1)^{-n}} \mathrm{d}x\right)$$

where $(e^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}\eta/\xi^2) = 1$, ie $\eta = \xi^2$ if $\overline{\phi^2} \gg 1$. After integration the sums may be expanded in powers of $\xi^2/\overline{\phi^2}W_0^2$ which is small if $\overline{\phi^2} \gg 1$ and $W_0^2 \gtrsim \xi^2$. This leads to

$$A \simeq W_0^2[\exp(-\xi^2/W_0^2)+O(\xi^2/\overline{\phi^2}W_0^2)] \tag{A.10}$$

so that

$$I_1 = \frac{\pi^4 W_0^4 \xi^4 E_0^4}{2\overline{\phi^2}^2} e^{-\xi^2/W_0^2} \exp\left(-\frac{k^2\xi^2\sin^2\theta}{2\overline{\phi^2}}\right) \tag{A.11}$$

or, after normalization using equation (18),

$$I_1/\langle I\rangle^2 = 2\exp(-\xi^2/W_0^2). \tag{A.12}$$

Similarly

$$B = \sum_{n=0}^{\infty} (-1)^n \left(\int_0^\eta \frac{\exp[x(-1/2W_0^2+n\overline{\phi^2}/\xi^2)]}{(e^{\overline{\phi^2}}-1)^{n+1}} \mathrm{d}x + \int_\eta^\infty \frac{\exp\{-x[1/2W_0^2+(n+1)\overline{\phi^2}/\xi^2]\}}{(e^{\overline{\phi^2}}-1)^{-n}} \mathrm{d}x\right)$$

$$\simeq 2W_0^2 e^{-\overline{\phi^2}}(1-e^{-\xi^2/2W_0^2})+O(\xi^2/\overline{\phi^2}W_0^2) \tag{A.13}$$

so that

$$I_2 = \frac{\pi^4 W_0^6 \xi^2 E_0^4}{4\overline{\phi^2}}(1-e^{-\xi^2/2W_0^2})^2 \exp\left(-\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right) \tag{A.14}$$

and after normalization using (18) we obtain

$$\frac{I_2}{\langle I\rangle^2} = \frac{W_0^2\overline{\phi^2}}{\xi^2}(1-e^{-\xi^2/2W_0^2})^2 \exp\left(\frac{k^2\xi^2\sin^2\theta}{4\overline{\phi^2}}\right). \tag{A.15}$$

The second moment, equation (20), is just the sum of the two results (A.12) and (A.15).

A.2. Spatial coherence functions

When detection is carried out at two points, the term $\exp(2\mathrm{i}kr''\sin\theta\cos\eta)$ appearing on the right-hand sides of (A.1) and (A.2) is replaced by $\exp \mathrm{i}k(\boldsymbol{r}'''\,.\,\boldsymbol{v}-\boldsymbol{r}''\,.\,\boldsymbol{u})$ where $\boldsymbol{u}$ and $\boldsymbol{v}$

are defined by (23). The r'' integral can be performed exactly as before and the prefactors on the right-hand side of (A.3) and (A.4) are the same except that $\sin^2\theta$ becomes $u^2/4$. The factor $\exp(ik\mathbf{r}'''.\mathbf{v})$ which appears inside the integrals does not affect the term in $\exp(-2\overline{\phi^2}r'''^2/\xi^2)$ in (A.3) in the approximation (A.5) so that I_1 may be written

$$I_1 = \frac{\pi^4 W_0^2 \xi^4 E_0^4}{4\overline{\phi^2}^2} \exp\left(-\frac{k^2\xi^2u^2}{8\overline{\phi^2}}\right)(A+C) \tag{A.16}$$

where A is defined by equation (A.7) and

$$C = \int_0^\infty \frac{e^{-x/W_0^2} J_0(kv\sqrt{x})}{[1+(e^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}x/\xi^2)]^2}\,dx. \tag{A.17}$$

In the limit $\xi^2 \ll W_0^2$ the denominator is unity over virtually the whole range of integration so that

$$C = W_0^2 \exp(-k^2v^2W_0^2/4). \tag{A.18}$$

However, in general a correction factor will arise, by analogy with (A.10) of the form $\exp(-\xi^2/W_0^2) \simeq 1-\xi^2/W_0^2$.

The 'non-Gaussian' term I_2 takes the form

$$I_2 = \frac{\pi^4 W_0^2 \xi^2 E_0^4\, e^{2\overline{\phi^2}}}{16\overline{\phi^2}} \exp\left(-\frac{k^2\xi^2u^2}{16\overline{\phi^2}}\right)D^2 \tag{A.19}$$

where

$$D = \int_0^\infty dx \frac{\exp\{-x[(1/2W_0^2)+(\overline{\phi^2}/\xi^2)]\}}{1+(e^{\overline{\phi^2}}-1)\exp(-\overline{\phi^2}x/\xi^2)} J_0(\tfrac{1}{2}kv\sqrt{x}) \tag{A.20}$$

which reduces to (A.9) when $v = 0$. (A.20) can be evaluated if we neglect the $1/2W_0^2$ in the exponent (ie in the limit $\xi^2 \ll W_0^2$). Integration by parts and a change of origin then lead to

$$D = \frac{\overline{\phi^2}}{kv\xi^2} e^{-\overline{\phi^2}} \int_{-\xi^2}^{\infty} (x+\xi^2)^{1/2} J_1(\tfrac{1}{2}kv(x+\xi^2)^{1/2})\,\mathrm{sech}^2\left(\frac{1}{2}\frac{\overline{\phi^2}}{\xi^2}x\right)dx. \tag{A.21}$$

The main contribution to the integral comes from the region $|x| < \xi^2/\overline{\phi^2} < \xi^2$ so that the lower limit may be extended to $-\infty$ (the errors thus incurred being of order $e^{-\overline{\phi^2}}$). An expansion of the type (Watson 1944, p 140)

$$(z+h)^{1/2} J_1((z+h)^{1/2}) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}h)^m}{m!} z^{\frac{1}{2}(1-m)} J_{1-m}(\sqrt{z}) \tag{A.22}$$

may then be used to obtain D in the form of a series of Bessel functions

$$D = \xi^2 e^{-\overline{\phi^2}}\left[\frac{2J_1(\frac{1}{2}kv\xi)}{\frac{1}{2}kv\xi} - \frac{2}{\frac{1}{2}kv\xi}\sum_{m=1}^{\infty}\left(1-\frac{1}{2^{2m-1}}\right)\zeta(2m)\left(\frac{kv\xi}{4\overline{\phi^2}}\right)^{2m} J_{2m-1}(\tfrac{1}{2}kv\xi)\right] \tag{A.23}$$

where $\zeta(2m)$ is the Riemann zeta function. The series converges very rapidly in typical situations for which $(kv\xi/4\overline{\phi^2}) < 1$, and only the first term on the right-hand side of (A.23) has been retained in the text. This could have been obtained immediately from (A.20) by observing that the integrand cuts off sharply at $x \sim \xi^2$, so that D could be written approximately

$$D = e^{-\overline{\phi^2}} \int_0^{\xi^2} dx\, J_0(\tfrac{1}{2}kv\sqrt{x}) = \xi^2 e^{-\overline{\phi^2}}\left(\frac{2J_1(\frac{1}{2}kv\xi)}{\frac{1}{2}kv\xi}\right), \qquad \tfrac{1}{2}kv\xi \ll 2\overline{\phi^2}. \tag{A.24}$$

A.3. Temporal coherence functions

When $\sigma(\tau)\overline{\phi^2} \gg 1$ the approximation (19) leads to integrals analogous to (A.1) and (A.2). In the second term of the numerator, and in the denominator of the integrand on the right-hand side of I_1, $\overline{\phi^2}$ is replaced by $\sigma(\tau)\overline{\phi^2}$ and this integral can be evaluated using the approach given in § A.1 above. I_2 takes the form

$$I_2 = \pi W_0^2 E_0^4 \exp(2\overline{\phi^2}\sigma(\tau)) \int_{-\infty}^{+\infty} \mathrm{d}^2r'\, \mathrm{d}^2r''\, \mathrm{d}^2r''' \exp[2ikr'' \sin\theta \cos\eta - (r'^2 + r''^2 + r'''^2)/W_0^2]$$
$$\times \frac{\exp[-\overline{\phi^2}(|r''+r'''|^2 + |r''-r'''|^2 + \sigma(\tau)|r'+r''|^2 + \sigma(\tau)|r'-r''|^2)/\xi^2]}{\{1+(\mathrm{e}^{\sigma(\tau)\overline{\phi^2}}-1)\exp[-\sigma(\tau)\overline{\phi^2}|r'-r'''|^2/\xi^2]\}\{1+(\mathrm{e}^{\sigma(\tau)\overline{\phi^2}}-1)\exp[-\sigma(\tau)\overline{\phi^2}|r'+r'''|^2/\xi^2]\}}. \tag{A.25}$$

After integration over r'' this reduces to

$$I_2 = \frac{\pi^2 W_0^2 \xi^2 E_0^4 \exp(2\overline{\phi^2}\sigma(\tau))}{2\overline{\phi^2}(1+\sigma(\tau))} \exp\left(-\frac{k^2\xi^2 \sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right)$$
$$\times \int_{-\infty}^{+\infty} \mathrm{d}^2r'\, \mathrm{d}^2r''' \frac{\exp[-2\overline{\phi^2}(r'''^2 + \sigma(\tau)r'^2)/\xi^2]\exp[-(r'^2+r'''^2)/W_0^2]}{\{\ldots\}\{\ldots\}}. \tag{A.26}$$

Two limiting cases may now be distinguished:

(i) $\sigma(\tau) \sim 1$. The integrand is the same as that appearing in (A.4) except that the denominator contains the product $\sigma(\tau)\overline{\phi^2}$ rather than simply $\overline{\phi^2}$. The treatment given in § A.1 can be followed and leads, for $\xi^2 \ll W_0^2$, to

$$I_2 = \frac{\pi^4 W_0^2 \xi^6 E_0^4}{8\overline{\phi^2}(1+\sigma(\tau))} \exp\left(-\frac{k^2\xi^2\sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right) \tag{A.27}$$

(ii) $\sigma(\tau) \ll 1$ (but $\sigma(\tau)\overline{\phi^2} \gg 1$). In this case the main contribution to the integral comes from the region $r''' \sim 0$ and we can use the approximation (A.5):

$$I_2 = \frac{\pi^4 W_0^2 \xi^4 E_0^4}{4\overline{\phi^2}^2(1+\sigma(\tau))} \exp(2\overline{\phi^2}\sigma(\tau)) \exp\left(-\frac{k^2\xi^2\sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right)$$
$$\times \int_0^\infty \mathrm{d}x \frac{\exp\{-x[(2\overline{\phi^2}\sigma(\tau)/\xi^2) + (1/W_0^2)]\}}{[1+(\mathrm{e}^{\overline{\phi^2}}-1)\,\mathrm{e}^{-\sigma\overline{\phi^2}x/\xi^2}]^2}. \tag{A.28}$$

If we neglect the $1/W_0^2$ in the exponent on the right-hand side, the integral may be evaluated to give ($\xi^2 \ll W_0^2$)

$$I_2 = \frac{\pi^4 W_0^2 \xi^6 E_0^4}{4\overline{\phi^2}^2(1+\sigma(\tau))} \exp\left(-\frac{k^2\xi^2\sin^2\theta}{2\overline{\phi^2}(1+\sigma(\tau))}\right) \tag{A.29}$$

which is seen to be at least a factor $4/\overline{\phi^2}$ smaller than (A.27) since $\sigma(\tau) \ll 1$.

Note added in proof. We are grateful to Professor M Bertolotti and Drs B Crosignani and P di Porto for pointing out to us that, strictly speaking, the quantity E_0 appearing in equation (7) should be angle-dependent. There is considerable debate in the literature as to the nature of such a variation which depends critically on the boundary conditions used to reduce the Helmholtz formula to relation (7). If the gradient of the phase variable with respect to the z direction (figure 2) is assumed to vanish at the emergent

$z = 0$ plane then $E_0 \propto 1+\cos\theta$ and at large angles the scattered intensity will fall off faster than predicted by (18). However, since E_0 cancels from the formulae for the normalized statistical and correlation properties of the radiation, these will be unaffected.

References

Beckmann P 1973 *IEEE Trans. Antennae Propag.* **21** 169–75
Beckmann P and Spizzichino A 1963 *The Scattering of Electromagnetic Waves from Rough Surfaces* (Oxford: Pergamon)
Berry M V 1973 *Phil. Trans. R. Soc.* **273** 611–54
Cox C and Munk W 1954 *J. Mar. Res.* **13** 198–227
Davenport W B and Root W L 1958 *An Introduction to the Theory of Random Signals and Noise* (New York: McGraw-Hill)
Deutsch Ch and Keating P N 1969 *J. Appl. Opt.* **40** 4049–54
Enloe L H 1967 *Bell Syst. Tech. J.* **46** 1479–89
Erdelyi A (ed.) 1954 *Tables of Integral Transforms* vol 1 (New York: McGraw-Hill)
Estes L E, Narducci L M and Tuft R A 1971 *J. Opt. Soc. Am.* **61** 1301–6
Goldstein H 1951 *Propagation of Short Radio Waves, Rad. Lab. Series* vol 13 ed D E Kerr (New York: McGraw-Hill)
Heilmeier G H, Zanoni L A and Barton L A 1968 *Proc. IEEE* **56** 1162–71
Jakeman E 1974 *Photon Correlation and Light Beating Spectroscopy* eds H Z Cummins and E R Pike (New York: Plenum)
Jakeman E and Pusey P N 1973a *J. Phys.* A: *Math., Nucl. Gen.* **6** L88–92
—— 1973b *Phys. Lett.* **44A** 456–8
Kivelson M G and Moszkowski S A 1965 *J. Appl. Phys.* **36** 3609–12
Mandel L and Wolf E 1965 *Rev. Mod. Phys.* **37** 231–87
Marathay A S, Heiko L and Zuckermann J L 1970 *Appl. Opt.* **9** 2470–6
Mercier R P 1962 *Proc. Camb. Phil. Soc.* A **58** 382–400
Pusey P N, Schaefer D W and Koppel D E 1974 *J. Phys.* A: *Math., Nucl. Gen.* **7** 530–40
Rayleigh Lord (J W Strutt) 1919 *Phil. Mag.* **6** 321–47
Schaefer D W 1974 *Laser Applications to Optics and Spectroscopy* eds S F Jacobs, M O Scully, M Sargent and J F Scott (New York: Addison-Wesley)
Siegert A J F 1943 *MIT Rad. Lab. Rep.* No 465
Spragg R C and Whitehouse D J 1971 *Proc. Inst. Mech. Engrs.* **185** 697–707
Sprague R A 1972 *Appl. Opt.* **11** 2811–6
Watson G N 1944 *Theory of Bessel Functions* (London: Cambridge University Press)
Zipfel G G and DeSanto J A 1972 *J. Math. Phys.* **13** 1903–11